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On the use of asymptotic expansions*

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Divergent asymptotic expansions in quantum chemistry often must be evaluated on Stokes lines, where the form of the expansion changes discontinuously and might appear to be ambiguous. Towards clarifying the use of asymptotic expansions on Stokes lines we discuss by numerical example the Airy function $Bi(x)$ for real, positive x. Two physical problems to which this example is relevant, among others, are the Rayleigh-Schrödinger perturbation theory for the LoSurdo-Stark effect in hydrogen and the JWKB connection-formula problem, for which real series are associated with complex sums. The various roles of partial summation, Padé approximants, and Borel summation are compared. In addition, a derivation is given for an integral that occurs in a simple proof of the Borel summability of asymptotic expansions for the confluent hypergeometric function, which function is fundamental to certain quantum chemistry problems, and which integral is given incorrectly in several standard references.

Key words: Asymptotic expansion — JWKB — Perturbation theory — Borel summation — LoSurdo-Stark effect

1. Introduction

A divergent asymptotic expansion is often the simplest form for solution of the Schrödinger equation. The Rayleigh-Schrödinger perturbation theory (RSPT) for hydrogen in an electrostatic field (the LoSurdo-Stark effect) is a prime example. In practice, the accuracy of partial sums of a divergent series is determined by the smallest term. If the solution is needed to higher numerical precision, more sophisticated "summation" methods must be used, such as (in the case of the LoSurdo-Stark effect) Padé approximants [1-3] and/or Borel summation [4].

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* Dedicated to Professor J. Koutecký on the occasion of his 65th birthday

Besides limited accuracy of partial sums, asymptotic expansions have a second complication: there are lines (in the complex plane) across which the coefficients of exponentially small terms in the expansion change discontinuously - so-called Stokes lines. Often the solution is needed for values of the physical variable on a Stokes line. The LoSurdo-Stark effect is again such a case, and the Jeffreys-Wentzel-Kramers-Brillouin (JWKB) method inside a barrier is a second [5, 6]. The question of how one evaluates the sum of a series where the series is undergoing a discontinuous change is a natural source for confusion. The possibility for confusion is compounded by the knowledge that sophisticated summation techniques can give complex sums to real series (on the Stokes lines) and real sums to complex series [5-8]. For example, the analytically continued Borel-Padé summation and the complex Padé-Padé methods applied to the *real* RSPT series in the LoSurdo-Stark effect both give (numerically) the correct *complex* resonance eigenvalue [3,4].

The aim of this article is to clarify two points about the use of asymptotic expansions (as they occur in the physical problems we have encountered). (i) The first is the summation of an expansion on its line of discontinuity: what we do here is to carry out illustrative numerical calculations by various methods for a specific, simple, prototypical example - the Airy Bi function - to demonstrate how to resolve the questions of accuracy and ambiguity. (ii) The second point is a more technical one. It has to do with a key step in a simple, direct proof [9] of the Borel-summability of the asymptotic expansions for the confluent hypergeometric function, special cases of which occur in several fundamental applications in quantum theory, as well as outside. In the derivation, there is a crucial integral that has been given incorrectly in many standard references, and the evaluation of which might be considered the only "difficult" part of the derivation. We evaluate that integral here in a straightforward way by using the Laplace-transform convolution theorem.

2. Numerical evaluation of the Airy $Bi(z)$ function from its asymptotic expansions

The focus of our numerical illustration is the Airy $Bi(z)$ function. In quantum mechanics, the Airy equation is the Schrödinger equation for motion in a uniform, constant field, and it appears in discussions of the JWKB method at a linear turning point. $Bi(z)$ is a solution of

$$\frac{d^2 Bi(z)}{dz^2} = z\, Bi(z) \tag{1}$$

that grows exponentially as $z \to \infty$. However, the positive real axis, $z > 0$, is a Stokes line of the asymptotic expansion for $Bi(z)$.

Suppose one wants to evaluate $Bi(z)$ for $z > 0$ from its asymptotic expansion. There are three asymptotic expansions for $|z|$ large in the right half-plane given in standard reference works [10]:

$$Bi(z) \sim \pi^{-1/2} z^{-1/4}\, e^{\zeta} \sum_{k=0}^{\infty} c_k \zeta^{-k} - i\tfrac{1}{2}\pi^{-1/2} z^{-1/4}\, e^{-\zeta} \sum_{k=0}^{\infty} c_k (-\zeta)^{-k},$$

$$[-\pi < \arg(z) < \pi/2 \text{ (sense of Poincaré)}, \quad -\tfrac{2}{3}\pi < \arg(z) < 0 \text{ (sense of Borel)}], \tag{2}$$

$$Bi(z) \sim \pi^{-1/2} z^{-1/4}\, e^{\zeta} \sum_{k=0}^{\infty} c_k \zeta^{-k} + i\tfrac{1}{2}\pi^{-1/2} z^{-1/4} e^{-\zeta} \sum_{k=0}^{\infty} c_k (-\zeta)^{-k},$$

$$[-\pi/2 < \arg(z) < \pi \text{ (sense of Poincaré)}, \quad 0 < \arg(z) < \tfrac{2}{3}\pi \text{ (sense of Borel]}, \tag{3}$$

$$Bi(z) \sim \pi^{-1/2} z^{-1/4}\, e^{\zeta} \sum_{k=0}^{\infty} c_k \zeta^{-k}, [-\pi/3 < \arg(z) < \pi/3 \text{ (sense of Poincaré)}]. \tag{4}$$

The coefficients c_k, which grow like $(k-1)!/(2^{k+1}\pi)$ for large k, are given by

$$c_k = \frac{\Gamma(k+\frac{5}{6})\Gamma(k+\frac{1}{6})}{\Gamma(\frac{5}{6})\Gamma(\frac{1}{6})2^k k!} \sim \frac{(k-1)!}{2^{k+1}\pi}. \tag{5}$$

The parameter ζ is

$$\zeta = \tfrac{2}{3} z^{3/2}. \tag{6}$$

In most references, the domains of applicability that are given are respectively, $-\pi < \arg(z) < \pi/2$, $-\pi/2 < \arg(z) < \pi$, and $-\pi/3 < \arg(z) < \pi/3$ (the ones indicated above by "sense of Poincaré"). All three would appear to be valid for $z > 0 [\arg(z) = 0]$. The question we address is, *which of the three expansions should be used to evaluate* $Bi(z)$ *on the positive real axis,* $z > 0$?

The expansions differ in the coefficients of the exponentially small terms ($e^{-\zeta}$), which are $+i$, $-i$, and 0. The Poincaré definition, that $\sum_{k=0}^{\infty} a_k x^k$ is an asymptotic power series for $f(x)$ if

$$\left| f(x) - \sum_{k=0}^{n} a_k x^k \right| = O(|x|^{n+1}), \quad \text{as } x \to +0, \tag{7}$$

does not distinguish among exponentially small terms and includes the real axis for all three. The *numerical result* surely depends on which expansion is used.

From one point of view, the question is a practical one. One should recall that partial sums of asymptotic expansions, because of their divergent character, approximate a function with strictly limited accuracy. One should first specify the *method* to be used to *calculate* with the asymptotic expansion, then pick the expansion that give the *smallest error.* From this point of view, the error is always nonzero, and there is no unique choice of expansion independent of the calculational procedure. If partial summation is the method, then on $z > 0$ the real expansion (4) gives smaller errors in approximating $Bi(z)$, which is real for real

z, than (2) and (3), which differ from (4) only by imaginary terms. Thus for partial summation on $z>0$, the answer to the question is the real expansion (4). There are, however, other computational methods to be discussed below.

From a second point of view, the question is in part theoretical: to determine a procedure that uniquely recovers the function from the expansion (with zero error). One such procedure has been known for 88 years - the summation method of Borel [11]. Borel's method is applicable [9] to expansions (2) and (3) above to give $Bi(z)$ on the domains labeled "sense of Borel," but not to (4). That is, the series that gives the smallest errors for partial summation on the real axis is not Borel summable to $Bi(z)$. To use (4) with Borel summation would give the wrong answer. Note that the domains for Borel summability, $-\frac{2}{3}\pi<\arg(z)<0$, and $0<\arg(z)<\frac{2}{3}\pi$, are smaller than for the sense of Poincaré and are non-overlapping. The positive z axis, $\arg(z)=0$, would appear to be excluded from the standard definition of the Borel method applied to $Bi(z)$, but it is more or less trivially included by analytic continuation (to be elaborated on below). In fact there is a standard set of uniqueness theorems of Watson [12], Nevanlinna [13], and Carleman [14] attaching a specific analytic function to an asymptotic expansion, and for $Bi(z)$ the applicability of the Borel method for $Bi(z)$ implies these standard uniqueness theorems. Further, if some other summation method works, then either it must be consistent with these theorems, or it will imply a new (as yet undiscovered) uniqueness theorem. In contrast to the first point of view, at any particular value of z the expansion is unique depending only on $\arg(z)$, but independent of the calculational method. This applies to the boundary, $\arg(z)=0$, but the boundary must first be assigned either to the sector below or above. How this works computationally will be seen immediately below in the Padé-Padé approach. It should be emphasized that when computationally implemented, a summation technique like the Borel method is capable of giving the desired analytic function to zero error. This is particularly important if the only approach one has available for calculating the function is through the asymptotic expansion.

We illustrate these considerations with numerical calculations. We use the asymptotic expansions to compute $Bi(z)$ by three techniques and see which come out correct. The first computational technique is partial summation (the original sense of an asymptotic expansion). The second is Padé approximants [15], which require only a few lines of computer code, and which are often used heuristically to speed convergence or to induce convergence when there is divergence. (Here they do not work on the real $z>0$ axis.) For the third technique we might have used, in the light of the above discussion, Borel summation, as was implemented in [4]. However, we use instead a more pedestrian method that involves using Padé approximants twice [3], but that otherwise appears to give numerical results that are the same as the Borel method [4], which in turn is not surprising in view of the discussion in the preceding paragraph. The computer-programming effort is only a little greater than using Padé approximants once. The Padé-Padé method *as we implement it here* consists of (i) calculating the sums of the power series in Eqs. (2)-(4) *and their successive derivatives* with respect to ζ^{-1} at some intermediate complex value z_{int} (and thus ζ_{int}^{-1}) by Padé approximants to obtain the coefficients of the *convergent* power series in $(\zeta^{-1}-\zeta_{\text{int}}^{-1})$ for the functions whose Borel-summable asymptotic expansions are the power-series in Eqs. (2)-(4), followed by (ii) partial summation (or Padé summation, since the computer subroutine is already at hand) of these convergent power series in $(\zeta^{-1}-\zeta_{\text{int}}^{-1})$ at the desired final value of z, which in our case is on the real z (and consequently real ζ^{-1}) axis (i.e. analytic continuation). Multiplication by $z^{-1/4}e^{\pm\zeta}$ and combination of the results for the two subseries [in the cases of Eqs. (2) and (3)] give the Padé-Padé-based values for $Bi(z)$ using Eqs. (2)-(4). [We remark, but do not illustrate numerically, that the final value of z is not only permitted to be on the real axis, but it may even be complex on the opposite side of the real axis from z_{int}, so long as $|\zeta^{-1}-\zeta_{\text{int}}^{-1}|<|\zeta_{\text{int}}^{-1}|$.]

Now let us pick a value of z, say $z=2.5$, and compute $Bi(z)$. The results are displayed in Table 1.

Note first that the smallest term in the partial summations occurs at $n=6$. The best partial sum, obtained by stopping with $n=5$, is $Bi(2.5)\sim 6.48213\ldots$, which coincides with the exact result of 6.48166 ... to three significant figures. The error

Table 1. Calculation of $Bi(z)$[a] at a single point, $z=2.5$, from its asymptotic expansions by partial summation, Padé approximants, and analytically continued Padé-Padé approximants (about intermediate point $z_{\text{int}}=2-i$). The Padé-Padé approximants of the formally complex expansion, Eq. (2) of the text, are approaching the exact value of $Bi(2.5)$[a]

	Formally real expansion [Eq. (4)]			Formally complex expansion [Eq. (2)]
n	Partial sum	Padé approximant[b]	Padé-Padé approximant[b]	Padé-Padé approximant[b,c]
0	6.25758	6.25758	$6.25758+i0.00000$	$6.25758-i0.01609$
1	6.42248	6.42695	$6.42105+i0.00564$	$6.42106-i0.01002$
2	6.45594	6.46446	$6.48566+i0.02816$	$6.48565+i0.01245$
3	6.46894	6.47766	$6.50425+i0.04614$	$6.50424+i0.03041$
4	6.47642	6.49052	$6.47577+i0.03270$	$6.47577+i0.01697$
5	6.48213	6.53973	$6.48213+i0.01972$	$6.48213+i0.00400$
6	6.48758	6.46116	$6.48442+i0.02555$	$6.48441+i0.00983$
7	6·49380	6.48162	$6.47889+i0.02488$	$6.47889+i0.00915$
8	6.50209	6.50666	$6.48188+i0.01766$	$6.48188+i0.00193$
9	6.51469	6.44748	$6.48210+i0.01853$	$6.48210+i0.00281$
10	6.53624	6.48063	$6.48129+i0.01738$	$6.48129+i0.00166$
20	620.87316	6.44587	$6.48132+i0.01576$	$6.48132+i0.00003$
30	2.372×10^{9}	6.43092	$6.48165+i0.01573$	$6.48165+i0.00000$
40	3.131×10^{17}	6.46283	$6.48166+i0.01572$	$6.48166-i0.00001$

[a] The exact value of $Bi(2.5)$ is 6.48166 ...
[b] Let $[\frac{1}{2}n]$ denote the greatest integer $\leq\frac{1}{2}n$. Let $[N/M]$ denote the Padé approximant with numerator of degree N and denominator of degree M. The Padé approximants listed are $[[\frac{1}{2}n]/[\frac{1}{2}(n+1)]]$
[c] All coefficients of the divergent expansion through order n were used to generate the convergent series at the intermediate point $z_{\text{int}}=-i$

is 0.00047. After the 5th partial sum, the error in the partial sums increases factorially fast with n.

Note second that direct Padé approximants do not help. One cannot obtain with certainty even three significant figures from the results given in Table 1. It is perhaps significant that, in contrast to the partial sums, the Padé approximants stay close to the exact value of Bi, but they do not approach the accuracy of the 5th partial sum, let alone improve on it.

Off the real axis the situation is markedly different. Information in Table 2 pertains to $z=2-i$, [$\arg(z)=-\pi/6.77\ldots$], a value of z that still falls inside the Poincaré domains of all three expansions. Note first that the partial sums of the formally complex series (2) are significantly better than those of the real series (4). Otherwise they behave as at $z=2.5$, except that the term smallest in magnitude occurs at $n=5$ (vs 6). Note second that the Padé approximants, in marked contrast to the case for $z=2.5$, appear to converge for all three series. The converged value for the complex series (2) is the correct one for $Bi(2-i)$, while the value obtained from series (4) is the correct value for $Bi(2-i)+iAi(2-i)$, where $Ai(z)$ is the Airy function that decreases exponentially as $z\to+\infty$. (The value that would be obtained from series (3) is the correct value for $Bi(2-i)+2iAi(2-i)$.) If instead we had taken $z=2+i$, then (3) would have yielded the correct value, and (2) and (4) would have yielded $Bi(2+i)-2iAi(2+i)$ and $Bi(2+i)-iAi(2+i)$,

Table 2. Calculation of $Bi(z)$[a] at a single point, $z=2-i$, the intermediate point for Table 1, from its asymptotic expansions by partial summation and Padé approximants. The Padé approximants of the formally complex expansion, Eq. (2) of the text, are approaching the exact value of $Bi(2-i)$[a]

	Formally real expansion [Eq. (4)]		Formally complex expansion [Eq. (2)]	
n	Partial sum	Padé approximant[b]	Partial sum	Padé approximant[b]
0	$0.65277-i2.46974$	$0.65277-i2.46974$	$0.69441-i2.47085$	$0.69441-i2.47085$
1	$0.71768-i2.51579$	$0.72020-i2.51553$	$0.75831-i2.51770$	$0.76084-i2.51741$
2	$0.73671-i2.51429$	$0.73987-i2.50977$	$0.77740-i2.51589$	$0.78059-i2.51143$
3	$0.74297-i2.50816$	$0.74097-i2.50261$	$0.78374-i2.50990$	$0.78170-i2.50431$
4	$0.74358-i2.50223$	$0.73655-i2.50164$	$0.78425-i2.50393$	$0.77727-i2.50333$
5	$0.74056-i2.49776$	$0.73666-i2.50430$	$0.78132-i2.49943$	$0.77738-i2.50600$
6	$0.73473-i2.49608$	$0.73823-i2.50359$	$0.77543-i2.49783$	$0.77895-i2.50529$
7	$0.72722-i2.49938$	$0.73741-i2.50290$	$0.76791-i2.50100$	$0.77813-i2.50460$
8	$0.72148-i2.51094$	$0.73729-i2.50365$	$0.76232-i2.51270$	$0.77801-i2.50535$
9	$0.72687-i2.53350$	$0.73775-i2.50343$	$0.76734-i2.53524$	$0.77847-i2.50513$
10	$0.76447-i2.56154$	$0.73742-i2.50326$	$0.80554-i2.56283$	$0.77813-i2.50496$
15	$-6.08703+i1.96952$	$0.73752-i2.50336$	$-6.01145+i2.04351$	$0.77824-i2.50506$
20		$0.73752-i2.50339$		$0.77823-i2.50509$
25		$0.73751-i2.50339$		$0.77823-i2.50509$
30		$0.73751-i2.50340$		$0.77823-i2.50510$

[a] The exact value of $Bi(2-i)$ to six digits is $0.77823-i2.50510$

[b] Let $[\frac{1}{2}n]$ denote the greatest integer $\leq\frac{1}{2}n$. Let $[N/M]$ denote the Padé approximant with numerator of degree N and denominator of degree M. The Padé approximants listed are $[[\frac{1}{2}n]/[\frac{1}{2}(n+1)]]$

respectively. This is to emphasize that (2) is the appropriate expansion for $Bi(z)$ when $-\frac{2}{3}\pi < \arg(z) < 0$, that (3) is appropriate when $0 < \arg(z) < \frac{2}{3}\pi$, and that $\arg(z) = 0$ is the *boundary* between the two regions.

Finally, we return to the last two columns in Table 1. Here the Padé approximants have been used to evaluate the power series and its successive derivatives at an intermediate point, arbitrarily taken to be $z = 2 - i$, to generate a new power series at that point. This new power series was then summed at the real point $z = 2.5$ (by Padé summation, only because it was convenient - partial summation of this new power series would have sufficed). The results for the formally real expansion (4) give not $Bi(2.5)$ but $Bi(2.5) + iAi(2.5)$, while those for the formally complex expansion (2) give $Bi(2.5)$. This clearly shows by direct computation how the formally complex expansion (2) gives the correct real value for $Bi(z)$ on the real axis by a procedure that involves continuation to the real axis from below.

Thus these numerical computations provide the following "answer" to the question of which expansion to use for $Bi(z)$ on $z > 0$: The unique expansions associated with $Bi(z)$ are the formally complex ones, (2) and (3). The positive real axis $z > 0$ is a boundary between the two. If the computational technique goes beyond partial summation, is capable of zero-error, and implies analytic continuation (explicitly or implicitly) from below, as in the case here with the Padé-Padé method or with the Borel method, then use the formally complex expansion (2). From above, use expansion (3). If $Bi(z)$ is to be only approximated by partial summation, for which the error is roughly the size of the first omitted term in the series, then the number of imaginary terms that should be kept is zero, and the real expansion (4), gives the best approximation by partial summation on the real axis.

[We have not explored here the details of the transition from the real axis to the complex plane. That question has been examined for *partial summation* in a related case by Olver [16]. The main picture inferred from Olver's work is that the error in partial sums would be least for the real expansion (4) in a domain whose shape is approximately parabolic, surrounding the positive real z axis. Beyond that domain, the complex expansions (2) and (3) yield smaller errors for partial sums.]

3. Evaluation of an integral related to the Borel summability of asymptotic expansions for the confluent hypergeometric function

The "recipe" for the Borel sum of a power series $\sum_{k=0}^{\infty} a_k x^k$ can be summarized by

$$(\text{Borel Sum of } \textstyle\sum a_k x^k) = \int_0^{\infty} e^{-t}\left(\sum_{k=0}^{\infty} a_k (xt)^k / k!\right) dt. \tag{8}$$

If the summation on the right-hand-side has an analytic continuation outside its circle of convergence to a neighborhood of the real axis, and if the integral converges, then it is called the Borel sum of the series. With the expansion coefficients a_k appropriate for the confluent hypergeometric function, it is straightforward [9] to carry out the summation and integration implied by Eq. (8) - with one minor difficulty, the evaluation of an integral over a Gauss ${}_2F_1$ hypergeometric function in terms of a Whittaker confluent hypergeometric function $W_{b,m/2}$ (and a gamma function):

$$\int_0^{\infty} dt\, e^{-zt} t^{a-1} {}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}m-b, \tfrac{1}{2}-\tfrac{1}{2}m-b;\, a;\, -t) = z^{-a-b}\, e^{z/2}\Gamma(a)\, W_{b,m/2}(z). \tag{9}$$

The evaluation of this integral is by no means a new, unsolved problem. It is a known integral that can be looked up in tables [17]. The "problem" is that it is tabulated incorrectly in several places, and it is not obvious how to evaluate it directly to check which tabulation is correct. Moreover, if one probes standard reference works for a route to the evaluation of Eq. (9), then it is possible to encounter what for these authors is even more unfamiliar territory. See for instance Eqs. (6) and (10) of Sect. 5.2 of [18], in which Eq. (9) can be obtained via MacRobert's E-function generalization of the hypergeometric function.

In view of the importance of the confluent hypergeometric function in quantum-mechanical applications and the importance of the consequences of its Borel summability, it is of interest to have easily accessible a simple derivation of Eq. (9). The standard integral representation of the Whittaker function is [19]

$$W_{b,m/2}(z) = z^{(1+m)/2}\, e^{-z/2} \int_0^{\infty} e^{-zt} t^{(m-1)/2-b} (t+1)^{(m-1)/2+b}\, dt / \Gamma(\tfrac{1}{2}+\tfrac{1}{2}m-b). \tag{10}$$

If a direct, coordinate transformation to turn Eq. (10) into Eq. (9) exists, we have not found it. Nevertheless, the desired relation appears to be a simple consequence of the integral representation (10) for $W_{b,m/2}(z)$ and the Laplace transform convolution theorem [20]:

$$\left(\int_0^{\infty} e^{-zt} A(t)\, dt\right)\left(\int_0^{\infty} e^{-zt} B(t)\, dt\right) = \int_0^{\infty} e^{-zt}\left(\int_0^{t} A(s)B(t-s)\, ds\right) dt. \tag{11}$$

Take

$$A(t) = t^{(m-1)/2-b}(t+1)^{(m-1)/2+b} / \Gamma(\tfrac{1}{2}+\tfrac{1}{2}m-b) \tag{12}$$

which by Eq. (10) has $z^{-(1+m)/2}\, e^{+z/2}\, W_{b,m/2}(z)$ for its Laplace transform, and take

$$B(t) = t^{a-(1+m)/2+b-1}\Gamma(a)/\Gamma(a-\tfrac{1}{2}-\tfrac{1}{2}m+b), \tag{13}$$

whose Laplace transform is $z^{-a+(1+m)/2-b}\Gamma(a)$. The product of the Laplace transforms of A and B is the right-hand side of Eq. (9). The convolution of $A(t)$ and $B(t)$ is just t^{a-1} times the standard integral representation of ${}_2F_1$ [10]:

$$\int_0^{t} A(s)B(t-s)\, ds$$

$$= \frac{\Gamma(a)}{\Gamma(\frac{1}{2}+\frac{1}{2}m-b)\Gamma(a-\frac{1}{2}-\frac{1}{2}m+b)}$$

$$\times \int_0^1 ds\, s^{(m+1)/2-b-1}(1-s)^{a-(1+m)/2+b-1}(1-(-t)s)^{(m-1)/2+b}, \tag{14}$$

$$= t^{a-1}\,{}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}m-b, \tfrac{1}{2}-\tfrac{1}{2}m-b; a; -t). \tag{15}$$

Thus, by virtue of the convolution (15), the convolution theorem (11), and the product of the Laplace transforms of A and B, one finds that Eq. (9) is verified.

4. Concluding remarks

Although a Borel-summable asymptotic power series may be divergent, it is in one-to-one correspondence with an analytic function. By implementing numerically the Borel procedure, or by using other procedures such as Padé approximants and the analytically continued Padé-Padé methods illustrated here, the sum of the divergent series can be obtained to much higher accuracy than is permitted by partial summation. At the Stokes lines of the series, the form of the asymptotic expansion changes in the weighting of exponentially small contributions. As one approaches a Stokes line, Padé approximants that seem to converge away from a Stokes line become indecisive. By obtaining the function and its derivatives away from the Stokes line by Padé approximants of the divergent expansion, the function can be evaluated on the Stokes line numerically by summing the convergent power series back at the Stokes line (analytic continuation). The asymptotic expansion that gives the correct answer on the Stokes line by this procedure is the Borel summable one [Eq. (2) or (3), but not (4), for the Bi example]. If *partial sums* suffice, the most accurate partial sums are obtained by switching to the average of the two expressions [Eq. (4) for the Bi example] in an approximately quadratically shaped neighborhood of the Stokes line, as elucidated in particular by Olver [16].

The confluent hypergeometric functions have Borel-summable asymptotic expansions. A pedestrian proof of the summability of the series can be achieved by carrying out the procedure analytically. The only difficulty is to evaluate an integral that is not particularly elementary and that is unreliably reported in some tables of integrals. Towards clarification, we point out (with no claim of priority) that the integral follows in an elementary way from the Laplace transform convolution theorem.

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